

A twisted integrable hierarchy with \mathbb{D}_2 symmetry

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Abstract

A loop algebra approach to the Gerdjikov-Mikhailov-Valchev (GMV) equation is provided to exploit the associated twisted integrable structure and a new twisted integrable hierarchy is discovered. Using the twisted loop algebra structure, we obtain a transparent treatment of the associated scattering and inverse scattering theory and solve the initial value problem for the GMV equation.

1 Introduction

Symmetry is a novel phenomenon in Nature. It is also an important tools for scientists to unravel complicated dynamics. In the theory of integrable systems, studying symmetries has been one of the central problems and yields rewarding results even beyond the field itself. Roughly speaking, two main approaches are adopted: (1) try to understand specific problems by identifying simple, symmetric structures that lie within them [6], [21], etc; (2) try to classify integrable systems to provide a framework for ordering or understanding more general situation [17], [5], [19], etc.

One successful attempt in classification theory is the study of reduction groups, formulated in [16], [17], and developed in [18], [20], [8], [7]. Recent results have characterized Lax pairs with finite reduction groups of fractional-linear transformations, i.e., \mathbb{Z}_N , \mathbb{D}_N , \mathbb{T} , \mathbb{O} and \mathbb{I} and aroused interest in the classification theory of automorphic Lie algebras [12], [13], [14].

Despite progress made in the classification theory of algebraic structures, the analytic properties such as the construction of solutions, the investigation of the inverse

scattering theory, of the above integrable systems remain mostly open. In particular, one of the simplest systems with \mathbb{D}_2 -symmetry is the anisotropic deformation of a multicomponent generalization of the classical Heisenberg ferromagnetic equation:

$$\begin{aligned} i\vec{u}_t &= (\vec{u}_x - \vec{u}(\vec{u}^* \cdot \vec{u}_x))_x + 4\epsilon\vec{u}(\vec{u}^* \cdot J\vec{u}) + \mathbf{A}\vec{u}, \\ \vec{u}^*\vec{u} &= 1, \quad \vec{u} \in \mathbb{C}^{N-1}, \quad J^2 = 1, \quad [\mathbf{A}, J] = 0, \quad \epsilon > 0. \end{aligned} \quad (1.1)$$

Many interesting algebraic and analytic properties of (1.1) are provided in [10] but a complete resolution of the inverse problem and Cauchy problem is still demanded.

On the other hand, in studying symmetries of the generalized sine-Gordon equations (GSGE), famous for being connected to submanifold geometry in Euclidean spaces, Terng introduced twisted U/K -hierarchies via a loop group approach [22]. The inverse scattering problem of one prototypical class of twisted U/K -hierarchies is then solved by encoding the loop algebra structures into the inverse scattering theory of GSGE [1] and associated submanifold geometry in Minkowski spaces is derived [15]. Twisted U/K -hierarchies are integrable hierarchies with \mathbb{D}_2 symmetry.

Compared to the study of reduction groups, the loop group approach puts more emphasis on an organic assembling of ingredients of symmetries in integrable systems [2], [24], [23], [22], [15]. To illustrate, given three involutions τ , σ_0 and σ_1 on a simple Lie group, let U be the real form of τ , U/K be the symmetric space defined by σ_0 , $\mathcal{U} = \mathcal{K} + \mathcal{P}$, and $(\mathcal{L}_+, \mathcal{L}_-)$ be a splitting of the loop algebra $\mathcal{L}(\mathcal{U}/\mathcal{K})$ such that

$$\sigma_0(\xi(-\lambda)) = \xi(\lambda), \quad \sigma_1(\xi(1/\lambda)) = \xi(\lambda)$$

for $\xi \in \mathcal{L}_+$. Denote the corresponding projection map to \mathcal{L}_\pm as $\hat{\pi}_\pm$. A twisted U/K -hierarchy is then defined by the Lax pair

$$[\partial_x + \hat{\pi}_+(mJ_1m^{-1}), \partial_t + \hat{\pi}_+(mJ_jm^{-1})] = 0, \quad (1.2)$$

for some $m = m(x, t, \lambda) \in L_-$, the loop group corresponding to the loop algebra \mathcal{L}_- , and constant loops $J_j \in \mathcal{L}_+$ with coefficients in a Cartan subalgebra in \mathcal{P} . The loop group approach is rooted in, enhanced and enriched by the inverse scattering theory.

The purpose of this paper is to provide a loop algebra approach to the anisotropic deformation of the multicomponent generalization of the Heisenberg ferromagnetic equation (1.1), $\mathbf{N}=3$, called **the GMV equation** for simplicity from now on, and to solve the inverse scattering theory. Distinct features discovered are:

- The loop algebra factorization $(\mathcal{L}_+, \mathcal{L}_-)$ is not of a splitting type. Thus the twisted hierarchy associated with the GMV equation, i.e., the twisted $\frac{U(3)}{U(1) \times U(2)}$ -hierarchy, generalizes the twisted U/K -hierarchies defined in [22].
- One needs to introduce an extended spectral problem and extended scattering data to solve the inverse problem. The extended spectral problem chosen is that for the twisted $\frac{U(4)}{U(2) \times U(2)}$ -hierarchy which shares the same reduction group and can be "projected" to a twisted $\frac{U(3)}{U(1) \times U(2)}$ -spectral problem when the scattering data is an extended one.

The paper is organized as follows: in Section 2, we define the twisted $\frac{U(3)}{U(1) \times U(2)}$ -hierarchy via a non-split factorization of the loop algebra and compute the explicit formula of a decisive coefficient, for the GMV equation, in the Lax pair. Section 3 is the discussion of the GMV equation and its relation with the twisted $\frac{U(3)}{U(1) \times U(2)}$ -hierarchy. Section 4 and 5 are devoted to the scattering and inverse scattering theory of the twisted $\frac{U(3)}{U(1) \times U(2)}$ -hierarchy. The Cauchy problems of twisted $\frac{U(3)}{U(1) \times U(2)}$ -flows and the GMV equation are solved in Section 6.

We make two special remarks at last. Though the discussion in Section 2 and 3 should be extended for general N by analogy, the spectral problem for (1.1) is no longer defined by an oblique direction [1], [15] and the associated direct problem cannot be solved when $N > 3$. The other remark is a Bäcklund transformation theory for the twisted $\frac{U(4)}{U(2) \times U(2)}$ -flows should be obtained by adapting the amazing computation and theory for the GSGE [4]. However, the extended scattering data is not preserved under these transformations. So the approach yields no GMV solitons.

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2 The twisted $\frac{U(3)}{U(1) \times U(2)}$ -hierarchy

Let σ_i , $i = 1, 2$, be involutions on $U(3)$ defined by

$$\begin{aligned} \sigma_i(x) &= J_i x J_i^{-1}, \quad x \in U(3), \\ J_1 &= \text{diag}(1, -1, -1), \quad J_2 = \text{diag}(1, -1, 1) \end{aligned} \quad (2.1)$$

and $u(3) = \mathcal{K}_i \oplus \mathcal{P}_i$, $i = 1, 2$, the Cartan decompositions for σ_i . Let \mathcal{K}_i be the Lie algebras of K_i , i.e.,

$$\begin{aligned} K_1 &= \left\{ \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix} : |a_{11}| = 1, \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} \in U(2) \right\}, \\ K_2 &= \left\{ \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{pmatrix} : |a_{22}| = 1, \begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix} \in U(2) \right\}, \\ \mathcal{P}_1 &= \left\{ i \begin{pmatrix} 0 & u & v \\ u^* & 0 & 0 \\ v^* & 0 & 0 \end{pmatrix} \in u(3) \right\}, \quad \mathcal{P}_2 = \left\{ i \begin{pmatrix} 0 & u & 0 \\ u^* & 0 & v \\ 0 & v^* & 0 \end{pmatrix} \in u(3) \right\}. \end{aligned}$$

Hence

$$\begin{aligned} S &= K_1 \cap K_2 = \{ \text{diag} (e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3}) \mid \alpha_i \in \mathbb{R} \}, \\ \mathcal{S} &= \{ i \text{diag} (\alpha_1, \alpha_2, \alpha_3) \mid \alpha_i \in \mathbb{R} \}, \end{aligned} \quad (2.2)$$

and

$$K_1 = S \times_{S_1} K'_1, \quad K'_1 = 1 \otimes SU(2), \quad (2.3)$$

$$\mathcal{K}_1 = \mathcal{S} +_{S_1} \mathcal{K}'_1, \quad \mathcal{K}'_1 = 0 \oplus su(2), \quad (2.4)$$

$$S_1 = S \cap K'_1 = \{ \text{diag} (1, e^{i\alpha}, e^{-i\alpha}) \mid \alpha \in \mathbb{R} \}, \quad (2.5)$$

$$\mathcal{S}_1 = \mathcal{S} \cap \mathcal{K}'_1 = \{ i \text{ diag} (0, \alpha, -\alpha) \mid \alpha \in \mathbb{R} \}. \quad (2.6)$$

Here $K_1 = S \times_{S_1} K'_1$ means for $\forall x \in K_1$, $x = \xi\eta$ with $\xi \in S$, $\eta \in K'_1$ and if

$$x = \xi\eta = \tilde{\xi}\tilde{\eta}, \quad \xi, \tilde{\xi} \in S, \quad \eta, \tilde{\eta} \in K'_1,$$

then $\xi^{-1}\tilde{\xi} = \eta\tilde{\eta}^{-1} \in S_1$. By analogy $\mathcal{K}_1 = \mathcal{S} +_{S_1} \mathcal{K}'_1$ is defined by factorizations of elements in \mathcal{K}_1 up to factors in \mathcal{S}_1 .

Furthermore, for a fixed $\epsilon > 0$, define the loop groups

$$\begin{aligned} L^\epsilon &= \{ f : \mathfrak{A}_{\sqrt{\epsilon}\delta, \sqrt{\epsilon}/\delta} \xrightarrow{holo.} GL_3(\mathbb{C}) \mid (f(\bar{\lambda}))^* f(\lambda) = I, \sigma_1(f(-\lambda)) = f(\lambda) \} \\ L_+^\epsilon &= \{ f \in L \mid \sigma_2(f(\epsilon/\lambda)) = f(\lambda) \}, \\ L_-^\epsilon &= \{ f \in L \mid f : \mathbb{C}/\mathfrak{D}_{\sqrt{\epsilon}\delta} \xrightarrow{holo.} GL_3(\mathbb{C}), f(\infty) \in K'_1 \}. \end{aligned}$$

Here $0 < \delta < 1$, \mathfrak{S}^r is the circle of radius r centered at 0, \mathfrak{D}_r is the disk of radius r , and \mathfrak{A}_{r_1, r_2} is the annulus with boundaries \mathfrak{S}^{r_1} and \mathfrak{S}^{r_2} . Then the Lie algebras of L^ϵ , L_+^ϵ , L_-^ϵ are

$$\begin{aligned} \mathcal{L}^\epsilon &= \{ \xi(\lambda) = \sum_{j \leq n_0} \xi_j \lambda^j \mid \xi_j \in \mathcal{K}_1 \text{ if } j \text{ is even, } \xi_j \in \mathcal{P}_1 \text{ if } j \text{ is odd} \}, \\ \mathcal{L}_+^\epsilon &= \{ \xi(\lambda) = \sum_{|j| \leq n_0} \xi_j \lambda^j \in \mathcal{L}^\epsilon \mid \xi_{-j} = \sigma_2(\xi_j) \epsilon^j, \xi_0 \in \mathcal{S} \}, \\ \mathcal{L}_-^\epsilon &= \{ \xi(\lambda) = \sum_{j \leq 0} \xi_j \lambda^j \in \mathcal{L}^\epsilon \mid \xi_0 \in \mathcal{K}'_1 \}. \end{aligned}$$

Similarly, we have a non-splitting decomposition $\mathcal{L}^\epsilon = \mathcal{L}_+^\epsilon +_{S_1} \mathcal{L}_-^\epsilon$ and can define projections $\hat{\pi}_\pm$ of $\xi \in \mathcal{L}^\epsilon$ onto \mathcal{L}_+^ϵ , \mathcal{L}_-^ϵ , up to factors in \mathcal{S}_1 , by the following relations:

$$\begin{aligned} \hat{\pi}_+(\xi) &= \pi_{\mathcal{S}}(\xi_0) + \sum_{0 < j \leq n_0} \left(\xi_j \lambda^j + \sigma_2(\xi_j) \left(\frac{\epsilon}{\lambda} \right)^j \right), \\ \hat{\pi}_-(\xi) &= \pi_{\mathcal{K}'_1}(\xi_0) + \sum_{0 < j \leq n_0} (\xi_{-j} - \sigma_2(\xi_j) \epsilon^j) \lambda^{-j}, \\ \xi &= \hat{\pi}_+(\xi) +_{S_1} \hat{\pi}_-(\xi), \quad \xi_0 = \pi_{\mathcal{S}}(\xi_0) +_{S_1} \pi_{\mathcal{K}'_1}(\xi_0). \end{aligned} \quad (2.7)$$

Let $\xi(\lambda) \sim_{S_1} \tilde{\xi}(\lambda)$ mean that $\xi(\lambda) - \tilde{\xi}(\lambda)$ is a constant loop in \mathcal{S}_1 . Finally, let

$$\mathcal{A} = \{ i \begin{pmatrix} d_1 & r & 0 \\ r & d_1 & 0 \\ 0 & 0 & d_3 \end{pmatrix} \in u(3) : r, d_1, d_3 \in \mathbb{R} \} \quad (2.8)$$

be a maximal abelian subalgebra in $u(3)$,

$$a = i \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{P}_1 \cap \mathcal{A}, \quad (2.9)$$

and

$$\hat{J}_{1,0} = a\lambda + \sigma_2(a)\left(\frac{\epsilon}{\lambda}\right) \in \mathcal{P}_1 \cap \mathcal{A} \cap \mathcal{L}_+^\epsilon, \quad (2.10)$$

$$\hat{J}_k = i^{k-1}a^k\lambda^k - i \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_1 & 0 \\ 0 & 0 & d_3 \end{pmatrix} + i^{k-1}\sigma_2(a^k)\left(\frac{\epsilon}{\lambda}\right)^k \in \mathcal{A} \cap \mathcal{L}_+^\epsilon, \quad (2.11)$$

for $k \in \{1, 2, \dots\}$. Thus we have the commutativity condition

$$[\hat{J}_{1,0}, \hat{J}_k] = 0. \quad (2.12)$$

Definition 1. The k -th twisted $\frac{U(3)}{U(1) \times U(2)}$ -flow, parametrized by (d_1, d_3) , in the twisted $\frac{U(3)}{U(1) \times U(2)}$ -hierarchy is the compatibility condition

$$[\mathbf{L}, \mathbf{M}] = 0, \quad (2.13)$$

where

$$\mathbf{L} = \partial_x - \frac{\partial \Psi}{\partial x} \Psi^{-1} = \partial_x - (\lambda bab^{-1} + \frac{\epsilon}{\lambda} \sigma_2(bab^{-1})), \quad (2.14)$$

$$\mathbf{M} = \partial_t - \frac{\partial \Psi}{\partial t} \Psi^{-1}, \quad (2.15)$$

$$\Psi(x, t, \lambda) = m(x, t, \lambda) e^{x\hat{J}_{1,0} + t\hat{J}_k}, \quad (2.16)$$

for some $m = m(x, t, \cdot) \in L_-^\epsilon$ and $b(x, t) = m(x, t, \infty) \in \mathfrak{P}_1$ or \mathfrak{P}_2 . Here

$$\mathfrak{P}_1 = \{f : R^2 \rightarrow K'_1 | f(\cdot, t) - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{S}, \forall t\}, \quad (2.17)$$

$$\mathfrak{P}_2 = \{f : R^2 \rightarrow K'_1 | f(\cdot, t) - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \in \mathbb{S}, \forall t\}, \quad (2.18)$$

and \mathbb{S} is the Schwartz space.

We remark that the inverse scattering theory derived in this report shows $m(x, t, \lambda)$ is determined by $\partial_x^j b(x, t)$. Hence the k -th twisted $\frac{U(3)}{U(1) \times U(2)}$ -flow is a system in $b(x, t)$

and $b(x, t)$ is also called the k -th twisted $\frac{U(3)}{U(1) \times U(2)}$ -flow if no ambiguity occurs. Theorem 6 will provide the existence theorem by solving the initial value problem of the k -th twisted $\frac{U(3)}{U(1) \times U(2)}$ -flows.

By the definition of twisted $\frac{U(3)}{U(1) \times U(2)}$ -flows, one has

$$\begin{aligned} \frac{\partial \Psi}{\partial x} \Psi^{-1} &= \left\{ \left[\frac{\partial m}{\partial x} + m \left(\lambda a + \frac{\epsilon}{\lambda} \sigma_2(a) \right) \right] e^{x \hat{J}_{1,0} + t \hat{J}_k} \right\} e^{-x \hat{J}_{1,0} - t \hat{J}_k} m^{-1} \\ &= \frac{\partial m}{\partial x} m^{-1} + m \left(\lambda a + \frac{\epsilon}{\lambda} \sigma_2(a) \right) m^{-1} \\ &\sim_{\mathcal{S}_1} \hat{\pi}_+ \left(m \hat{J}_{1,0} m^{-1} \right). \end{aligned}$$

Similarly, $\frac{\partial \Psi}{\partial t} \Psi^{-1} \in \mathcal{L}_+^\epsilon$ as well and

$$\frac{\partial \Psi}{\partial t} \Psi^{-1} = \sum_1^k \left(P_j \lambda^j + \sigma_2(P_j) \left(\frac{\epsilon}{\lambda} \right)^j \right) + P_0 \quad (2.19)$$

$$\begin{aligned} &= \frac{\partial m}{\partial t} m^{-1} + m \hat{J}_k m^{-1}, \\ &\sim_{\mathcal{S}_1} \hat{\pi}_+ \left(m \hat{J}_k m^{-1} \right) \end{aligned} \quad (2.20)$$

Thus the k -th twisted $\frac{U(3)}{U(1) \times U(2)}$ -flows satisfy

$$\begin{aligned} \mathbf{L} &\sim_{\mathcal{S}_1} \partial_x - \hat{\pi}_+ (m \hat{J}_{1,0} m^{-1}), \\ \mathbf{M} &\sim_{\mathcal{S}_1} \partial_t - \hat{\pi}_+ (m \hat{J}_k m^{-1}) \end{aligned} \quad (2.21)$$

which are similar to (1.2).

Lemma 2.1. *The first twisted $\frac{U(3)}{U(1) \times U(2)}$ -flow is the linear system*

$$\frac{\partial}{\partial x} (bab^{-1}) - \frac{\partial}{\partial t} (bab^{-1}) = \left[bab^{-1}, i \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix} \right]$$

where c_i are real constants.

Proof. Following (2.13), (2.14), and (2.19), we obtain

$$\frac{\partial}{\partial x} (bab^{-1}) - \frac{\partial}{\partial t} (bab^{-1}) - [bab^{-1}, P_0] = 0, \quad (2.22)$$

$$\frac{\partial}{\partial x} P_0 - [bab^{-1}, \epsilon \sigma_2(bab^{-1})] - [\epsilon \sigma_2(bab^{-1}), bab^{-1}] = 0. \quad (2.23)$$

Thus P_0 is independent of x (and λ). Therefore P_0 is constant by taking the limit of (2.20) when $x \rightarrow -\infty$, $\lambda \rightarrow \infty$, and $m(x = -\infty, t, \lambda = \infty) = b(x = -\infty, t) = 1$ or

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \quad \square$$

We proceed to characterizing the k -th twisted $\frac{U(3)}{U(1) \times U(2)}$ -flow, $k \geq 2$, by showing that the coefficients P_j , $j \neq 0$, defined by (2.19), of \mathbf{M} can be computed explicitly in terms of x -derivatives of entries of $b(x, t)$. As for P_0 , owing to the non-splitting factorization $\mathcal{L}^\epsilon = \mathcal{L}_+^\epsilon +_{\mathcal{S}_1} \mathcal{L}_-^\epsilon$ (up to factors in \mathcal{S}_1), only the first diagonal entry of P_0 can be computed explicitly (in terms of x -derivatives of entries of $b(x, t)$). The last two diagonal entries of P_0 are (inexplicit) functions in $\partial_x^s b(x, t)$ as we have remarked earlier. This phenomenon is distinct from that of twisted flows defined in [15], [22].

Lemma 2.2. *For the k -th twisted $\frac{U(3)}{U(1) \times U(2)}$ -flow,*

$$m(\partial_x - \hat{J}_{1,0})m^{-1} = \partial_x - (bab^{-1}\lambda + \sigma_2(bab^{-1})\frac{\epsilon}{\lambda}), \quad (2.24)$$

$$m(\partial_t - \hat{J}_k)m^{-1} = \partial_t - \sum_1^k \left(P_j \lambda^j + \sigma_2(P_j) \left(\frac{\epsilon}{\lambda} \right)^j \right) - P_0. \quad (2.25)$$

Here P_i are defined by (2.19).

Proof. By (2.14), we have

$$(\partial_x \Psi) \Psi^{-1} = bab^{-1}\lambda + \sigma_2(bab^{-1})\frac{\epsilon}{\lambda}.$$

So (2.16) implies

$$(\partial_x m + m\hat{J}_{1,0})m^{-1} = bab^{-1}\lambda + \sigma_2(bab^{-1})\frac{\epsilon}{\lambda},$$

which is equivalent to (2.24). The identity (2.25) can be proved similarly. \square

Lemma 2.3. *For the k -th twisted $\frac{U(3)}{U(1) \times U(2)}$ -flow,*

$$[\partial_x - bab^{-1}\lambda - \sigma_2(bab^{-1})\frac{\epsilon}{\lambda}, m\hat{J}_k m^{-1}] = 0. \quad (2.26)$$

Proof. Because \hat{J}_k are loops with constant coefficients, by (2.12), we have

$$[\partial_x - \hat{J}_{1,0}, \hat{J}_k] = 0,$$

which implies

$$[m(\partial_x - \hat{J}_{1,0})m^{-1}, m(\hat{J}_k)m^{-1}] = 0.$$

By (2.24), we derive (2.26). \square

Lemma 2.4. *The coefficients P_j , $j \neq 0$, and the first diagonal element of P_0 of \mathbf{M} are fixed functions of components of $\partial_x^s b$, $0 \leq s \leq k - j$.*

By the decomposition property (2.7), Lemma 2.3, and defining

$$m\hat{J}_k m^{-1} = \sum_1^k \left(P_j \lambda^j + \sigma_2(P_j) \left(\frac{\epsilon}{\lambda} \right)^j \right) + P_0 + \sum_{j=0}^{\infty} R_j \lambda^{-j}, \quad (2.27)$$

we obtain the recursive formula on P_j , $0 < j \leq k$:

$$\begin{aligned} P_k &= i^{k-1} b a^k b^{-1} \\ \partial_x P_k - [bab^{-1}, P_{k-1}] &= 0, \\ \partial_x P_{k-1} - [bab^{-1}, P_{k-2}] - [\epsilon \sigma_2(bab^{-1}), P_k] &= 0, \\ \vdots \\ \partial_x P_2 - [bab^{-1}, P_1] - [\epsilon \sigma_2(bab^{-1}), P_3] &= 0, \\ \partial_x P_1 - [bab^{-1}, P_0 + R_0] - [\epsilon \sigma_2(bab^{-1}), P_2] &= 0. \end{aligned} \quad (2.28)$$

Therefore, we can adapt the argument of the proof of Lemma 2.3 in [15] to prove Lemma 2.4. We skip the proof. Instead, we compute the case $k = 2$ for the purpose of solving the Cauchy problem of the GMV equation in this report.

Lemma 2.5. *Write*

$$b(x, t) = m(x, t, \infty) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & u & -\bar{v} \\ 0 & v & \bar{u} \end{pmatrix} \in \mathfrak{P}_1 \cup \mathfrak{P}_2, \quad \vec{u} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad (2.29)$$

then

$$P_1 = \begin{pmatrix} 0 & -((1 - \vec{u}\vec{u}^*)\vec{u}_x)^* \\ (1 - \vec{u}\vec{u}^*)\vec{u}_x & 0_{2 \times 2} \end{pmatrix} \quad (2.30)$$

for the second twisted $\frac{U(3)}{U(1) \times U(2)}$ -flow.

Proof. We first define $T = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$, so

$$T^{-1} a T = \begin{pmatrix} -i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}, \quad (2.31)$$

$$bT = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} u & -\bar{v} & \frac{1}{\sqrt{2}} u \\ \frac{1}{\sqrt{2}} v & \bar{u} & \frac{1}{\sqrt{2}} v \end{pmatrix}, \quad (bT)^{-1} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \bar{u} & \frac{1}{\sqrt{2}} \bar{v} \\ 0 & -v & u \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \bar{u} & \frac{1}{\sqrt{2}} \bar{v} \end{pmatrix}, \quad (2.32)$$

$$P_2 = -i \begin{pmatrix} 1 & 0 & 0 \\ 0 & |u|^2 & u\bar{v} \\ 0 & \bar{u}v & |v|^2 \end{pmatrix}, \quad \partial_x P_2 = -i \begin{pmatrix} 0 & 0 & 0 \\ 0 & |u|_x^2 & (u\bar{v})_x \\ 0 & (\bar{u}v)_x & |v|_x^2 \end{pmatrix}. \quad (2.33)$$

By (2.28), we have

$$[bab^{-1}, P_1] = \partial_x P_2. \quad (2.34)$$

Taking the conjugation $(bT)^{-1} \cdot (bT)$ on both sides of (2.34) and using (2.29), (2.31)-(2.33), we obtain

$$\left[\begin{pmatrix} -i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}, (bT)^{-1}P_1(bT) \right] = -\frac{i}{\sqrt{2}} \begin{pmatrix} 0 & \bar{u}\bar{v}_x - \bar{u}_x\bar{v} & 0 \\ uv_x - u_xv & 0 & uv_x - u_xv \\ 0 & \bar{u}\bar{v}_x - \bar{u}_x\bar{v} & 0 \end{pmatrix}.$$

Thus the off diagonal part of $(bT)^{-1}P_1(bT)$, denoted as $[(bT)^{-1}P_1(bT)]^o$, is

$$[(bT)^{-1}P_1(bT)]^o = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \bar{u}\bar{v}_x - \bar{u}_x\bar{v} & 0 \\ -(uv_x - u_xv) & 0 & uv_x - u_xv \\ 0 & -(\bar{u}\bar{v}_x - \bar{u}_x\bar{v}) & 0 \end{pmatrix}. \quad (2.35)$$

On the other hand, using the minimal polynomial of $(bT)^{-1}m\hat{J}_2m^{-1}(bT)$ is

$$(X + i(\lambda^2 + d_1 + \frac{\epsilon^2}{\lambda^2})I)(X + id_3I),$$

we obtain

$$\begin{aligned} & \left[\begin{pmatrix} -i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{pmatrix} \lambda^2 + (bT)^{-1}P_1(bT)\lambda + (bT)^{-1}(P_0 + R_0)m^{-1}(bT) + \dots \right. \\ & \left. + i(\lambda^2 + d_1 + \frac{\epsilon^2}{\lambda^2})I \right] \times \left[\begin{pmatrix} -i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{pmatrix} \lambda^2 + (bT)^{-1}P_1(bT)\lambda \right. \\ & \left. + (bT)^{-1}(P_0 + R_0)m^{-1}(bT) + \dots + id_3I \right] = 0. \end{aligned} \quad (2.36)$$

Equating the λ^3 -coefficient of (2.36) yields the diagonal part of $(bT)^{-1}P_1(bT)$, denoted as $[(bT)^{-1}P_1(bT)]^d$, which is 0. Therefore, $(bT)^{-1}P_1(bT) = [(bT)^{-1}P_1(bT)]^o$. Together with (2.32) and (2.35), we obtain

$$\begin{aligned} P_1 &= \begin{pmatrix} 0 & v(\bar{u}\bar{v}_x - \bar{u}_x\bar{v}) & -u(\bar{u}\bar{v}_x - \bar{u}_x\bar{v}) \\ -\bar{v}(uv_x - u_xv) & 0 & 0 \\ \bar{u}(uv_x - u_xv) & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -((1 - \vec{u}\vec{u}^*)\vec{u}_x)^* \\ (1 - \vec{u}\vec{u}^*)\vec{u}_x & 0_{2 \times 2} \end{pmatrix}. \end{aligned}$$

□

3 The GMV equation

In studying integrable systems with reductions, one of the simplest nontrivial systems introduced by Gerdjikov, Mikhailov, Valchev [10], [11], [9], is the anisotropic multicomponent generalization of the classical Heisenberg ferromagnetic equation:

$$i\vec{u}_t = (\vec{u}_x - \vec{u}(\vec{u}^* \cdot \vec{u}_x))_x + 4\epsilon\vec{u}(\vec{u}^* \cdot J\vec{u}) + \mathbf{A}\vec{u}, \quad (3.1)$$

where

$$\begin{aligned} \vec{u}^* \vec{u} &= 1, \quad \vec{u}(x, t) \in \mathbb{C}^2, \\ J &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \quad \alpha, \beta \in \mathbb{R}. \end{aligned} \quad (3.2)$$

The equation (3.1), called the GMV equation for simplicity, has a $\mathbb{Z}_2 \times \mathbb{Z}_2$ reduced Lax representation

$$[\mathbf{L}, \mathbf{M}] = 0, \quad (3.3)$$

$$\mathbf{L} = \partial_x - bab^{-1}\lambda - \sigma_2(bab^{-1})\frac{\epsilon}{\lambda}, \quad (3.4)$$

$$\mathbf{M} = \partial_t - iba^2b^{-1}\lambda^2 - p_1\lambda - p_0 - \sigma_2(p_1)\frac{\epsilon}{\lambda} - i\sigma_2(ba^2b^{-1})\left(\frac{\epsilon}{\lambda}\right)^2, \quad (3.5)$$

with

$$a = i \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{A}, \quad b(x, t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & u & -\bar{v} \\ 0 & v & \bar{u} \end{pmatrix} \in K'_1, \quad (3.6)$$

$$p_1(x, t) = \begin{pmatrix} 0 & -\vec{a}^* \\ \vec{a} & 0 \end{pmatrix} \in \mathcal{P}_1, \quad \vec{a} = (1 - \vec{u}\vec{u}^*)\vec{u}_x, \quad \vec{u} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad (3.7)$$

$$p_0 = -i \begin{pmatrix} -2\epsilon\vec{u}^*J\vec{u} & 0 \\ 0 & \epsilon(J\vec{u}\vec{u}^* + \vec{u}\vec{u}^*J) \end{pmatrix} - i \operatorname{diag}(0, \alpha, \beta) \in \mathcal{S}. \quad (3.8)$$

It is readily to see that $\mathbf{L} - \partial_x \in \mathcal{L}_+^\epsilon$, $\mathbf{M} - \partial_t \in \mathcal{L}_+^\epsilon$.

Lemma 3.1. *Suppose*

$$[\mathbf{L}, \mathbf{M}'] = 0, \quad (3.9)$$

where \mathbf{L} is defined by (3.4), (3.6), and $\mathbf{M}' - \partial_t \in \mathcal{L}_+^\epsilon$ with $-iba^2b^{-1}\lambda^2$ as its leading term. Then there exist real functions $\gamma(x, t)$, $\alpha_1(t)$, $\alpha_2(t)$ and $\alpha_3(t)$, such that

$$\mathbf{M}' = \partial_t - iba^2b^{-1}\lambda^2 - p'_1\lambda - p'_0 - \sigma_2(p'_1)\frac{\epsilon}{\lambda} - i\sigma_2(ba^2b^{-1})\left(\frac{\epsilon}{\lambda}\right)^2, \quad (3.10)$$

with

$$p'_1 - p_1 = \gamma bab^{-1} \in \mathcal{P}_1, \quad (3.11)$$

$$p'_0 + i \begin{pmatrix} -2\epsilon\vec{u}^*J\vec{u} & 0 \\ 0 & \epsilon(J\vec{u}\vec{u}^* + \vec{u}\vec{u}^*J) \end{pmatrix} = -i \operatorname{diag}(\alpha_1, \alpha_2, \alpha_3), \quad (3.12)$$

and p_1 being the coefficient of \mathbf{M} defined by (3.7). Moreover,

$$\begin{aligned} i\vec{u}_t &= (\vec{u}_x - \vec{u}(\vec{u}^* \cdot \vec{u}_x) + i\gamma\vec{u})_x + 4\epsilon\vec{u}(\vec{u}^* \cdot J\vec{u}) + \mathbf{A}'\vec{u}, \\ \mathbf{A}' &= \text{diag}(\alpha, \beta), \quad \alpha = \alpha_2(t) - \alpha_1(t), \quad \beta = \alpha_3(t) - \alpha_1(t). \end{aligned} \quad (3.13)$$

Proof. By assumption, one can set

$$\mathbf{L} = \partial_x - q_1\lambda - \sigma_2(q_1)\frac{\epsilon}{\lambda}, \quad (3.14)$$

$$\mathbf{M}' = \partial_t - p'_2\lambda^2 - p'_1\lambda - p'_0 - \sigma_2(p'_1)\frac{\epsilon}{\lambda} - \sigma_2(p'_2)(\frac{\epsilon}{\lambda})^2, \quad (3.15)$$

with

$$q_1 = bab^{-1}, \quad p'_2 = iba^2b^{-1}, \quad (3.16)$$

$$p'_1 = \begin{pmatrix} 0 & -\vec{\theta}^* \\ \vec{\theta} & 0 \end{pmatrix} \in \mathcal{P}_1, \quad \vec{\theta}(x, t) = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}, \quad (3.17)$$

$$p'_0 = -i \begin{pmatrix} -2\epsilon\vec{u}^*J\vec{u} & 0 \\ 0 & \epsilon(J\vec{u}\vec{u}^* + \vec{u}\vec{u}^*J) \end{pmatrix} - i \text{diag}(\alpha_1, \alpha_2, \alpha_3), \quad (3.18)$$

and $\alpha_i = \alpha_i(x, t) \in \mathbb{R}$. The compatibility condition (3.9) then yields

$$\partial_x p'_2 - [q_1, p'_1] = 0, \quad (3.19)$$

$$\partial_x p'_1 - \partial_t q_1 - [q_1, p'_0] - [\epsilon\sigma_2(q_1), p'_2] = 0, \quad (3.20)$$

$$\partial_x p'_0 - [q_1, \epsilon\sigma_2(p'_1)] - [\epsilon\sigma_2(q_1), p'_1] = 0. \quad (3.21)$$

Let $\vec{u} = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$. Then (3.19) implies

$$\bar{u}_1\theta_1 + \bar{u}_2\theta_2 + u_1\bar{\theta}_1 + u_2\bar{\theta}_2 = 0, \quad (3.22)$$

$$\partial_x(|u_1|^2) - u_1\bar{\theta}_1 - \bar{u}_1\theta_1 = 0, \quad (3.23)$$

$$\partial_x(u_1\bar{u}_2) - u_1\bar{\theta}_2 - \bar{u}_2\theta_1 = 0. \quad (3.24)$$

It is an under-determined linear system. One then obtains

$$\vec{\theta} = (1 - \vec{u}\vec{u}^*)\vec{u}_x + i\gamma(x, t)\vec{u}, \quad \gamma(x, t) \in \mathbb{R}. \quad (3.25)$$

On the other hand, (3.20), (3.22)-(3.24) imply

$$-\partial_x \vec{\theta} + i\partial_t \vec{u} - \mathbf{A}'\vec{u} - 2\epsilon\vec{u}\vec{u}^*J\vec{u} - 2\epsilon\vec{u}\vec{u}^*J\vec{u} = 0 \quad (3.26)$$

with $\mathbf{A}' = \text{diag}(\alpha(x, t), \beta(x, t))$, $\alpha = \alpha_2 - \alpha_1$, and $\beta = \alpha_3 - \alpha_1$. Hence we obtain

$$i\vec{u}_t = (\vec{u}_x - \vec{u}(\vec{u}^* \cdot \vec{u}_x) + i\gamma\vec{u})_x + 4\epsilon\vec{u}(\vec{u}^* \cdot J\vec{u}) + \mathbf{A}'\vec{u}, \quad (3.27)$$

by (3.25). Moreover, (3.21) is equivalent to

$$\epsilon^{-1} \partial_x \alpha_1 - 2 \partial_x (-|u_1|^2 + |u_2|^2) + 2 \vec{u}^* J \vec{\theta} + 2 \vec{\theta}^* J \vec{u} = 0, \quad (3.28)$$

$$\partial_x \mathbf{A}'' + J \left(\partial_x (\vec{u} \vec{u}^*) - (\vec{\theta} \vec{u}^* + \vec{u} \vec{\theta}^*) \right) + \left(\partial_x (\vec{u} \vec{u}^*) - (\vec{\theta} \vec{u}^* + \vec{u} \vec{\theta}^*) \right) J = 0, \quad (3.29)$$

with $\mathbf{A}'' = \epsilon^{-1} \text{diag} (\alpha_2(x, t), \alpha_3(x, t))$. Note (3.22) and (3.23) imply

$$-2 \partial_x (-|u_1|^2 + |u_2|^2) + 2 \vec{u}^* J \vec{\theta} + 2 \vec{\theta}^* J \vec{u} = 0.$$

Together with (3.28) yields $\alpha_1 = \alpha(t)$. Besides,

$$\begin{aligned} & \partial_x (\vec{u} \vec{u}^*) - (\vec{\theta} \vec{u}^* + \vec{u} \vec{\theta}^*) \\ = & \vec{u}_x \vec{u}^* - \vec{\theta} \vec{u}^* + \vec{u} \vec{u}_x^* - \vec{u} \vec{\theta}^* \\ = & \vec{u}_x \vec{u}^* - [(1 - \vec{u} \vec{u}^*) \vec{u}_x + i \gamma \vec{u}] \vec{u}^* + \vec{u} \vec{u}_x^* - \vec{u} [\vec{u}_x^* (1 - \vec{u} \vec{u}^*) - i \gamma \vec{u}^*] \\ = & \vec{u}_x \vec{u}^* - (1 - \vec{u} \vec{u}^*) \vec{u}_x \vec{u}^* + \vec{u} \vec{u}_x^* - \vec{u} \vec{u}_x^* (1 - \vec{u} \vec{u}^*) \\ = & \vec{u} \vec{u}^* \vec{u}_x \vec{u}^* + \vec{u} \vec{u}_x^* \vec{u} \\ = & \vec{u} (|\vec{u}|^2)_x \vec{u}^* \\ = & 0. \end{aligned} \quad (3.30)$$

Here we have used (3.25) and $b \in 1 \times SU(2)$. Combining (3.29) and (3.30), we obtained $\alpha_i = \alpha_i(t)$, $i = 2, 3$. \square

Given $\epsilon > 0$, for the GMV equation parametrized by $(4\epsilon, \beta)$ or $(\alpha, -4\epsilon)$ with arbitrary real constants α, β , we have

Theorem 1. Write the second twisted $\frac{U(3)}{U(1) \times U(2)}$ -flow $b(x, t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & u & -\bar{v} \\ 0 & v & \bar{u} \end{pmatrix}$, then $\vec{u} = \begin{pmatrix} u \\ v \end{pmatrix}$ satisfies the GMV equation

$$\begin{aligned} i \vec{u}_t &= (\vec{u}_x - \vec{u} (\vec{u}^* \cdot \vec{u}_x))_x + 4\epsilon \vec{u} (\vec{u}^* \cdot J \vec{u}) + \mathbf{A}' \vec{u}, \\ \mathbf{A}' &= \begin{pmatrix} 4\epsilon & 0 \\ 0 & 2\epsilon + d_3 - d_1 \end{pmatrix} \text{ or } \begin{pmatrix} -2\epsilon + d_3 - d_1 & 0 \\ 0 & -4\epsilon \end{pmatrix}. \end{aligned} \quad (3.31)$$

Proof. By Definition 1 and Lemma 3.1, the second twisted $\frac{U(3)}{U(1) \times U(2)}$ -flow satisfies

$$\begin{aligned} i \vec{u}_t &= (\vec{u}_x - \vec{u} (\vec{u}^* \cdot \vec{u}_x) + i \gamma \vec{u})_x + 4\epsilon \vec{u} (\vec{u}^* \cdot J \vec{u}) + \mathbf{A}' \vec{u}, \\ \mathbf{A}' &= \text{diag} (\alpha, \beta), \quad \alpha = \alpha_2(t) - \alpha_1(t), \quad \beta = \alpha_3(t) - \alpha_1(t), \end{aligned}$$

where $\gamma(x, t)$, $\alpha_i(t)$ are defined by (3.15), (3.17), (3.18), and (3.25). By Lemma 2.5, we conclude $\gamma(x, t) \equiv 0$. Hence the theorem reduces to showing α_i , $i \in \{1, 2, 3\}$,

are certain constants determined by d_1 and d_3 . This can be seen by taking the limit of (2.20) when $x \rightarrow -\infty$ and $\lambda \rightarrow \infty$ since α_i are independent of x . Moreover, $m(x, t, \lambda) \rightarrow b(x, t)$ as $\lambda \rightarrow \infty$ will be shown in Theorem 6. Thus we have

$$\begin{aligned}
& \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_1 & 0 \\ 0 & 0 & d_3 \end{pmatrix} \\
&= \lim_{x \rightarrow -\infty} b^{-1} \left[\begin{pmatrix} -2\epsilon \vec{u}^* J \vec{u} & 0 \\ 0 & \epsilon (J \vec{u} \vec{u}^* + \vec{u} \vec{u}^* J) \end{pmatrix} + \text{diag}(\alpha_1, \alpha_2, \alpha_3) \right] b \\
&= \begin{cases} \begin{pmatrix} 2\epsilon + \alpha_1 & 0 & 0 \\ 0 & -2\epsilon + \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}, & \text{if } b \in \mathfrak{P}_1, \\ \begin{pmatrix} -2\epsilon + \alpha_1 & 0 & 0 \\ 0 & 2\epsilon + \alpha_3 & 0 \\ 0 & 0 & \alpha_2 \end{pmatrix}, & \text{if } b \in \mathfrak{P}_2. \end{cases} \tag{3.32}
\end{aligned}$$

So α_i are constant and $(\alpha, \beta) = (4\epsilon, 2\epsilon + d_3 - d_1)$ or $(-2\epsilon + d_3 - d_1, -4\epsilon)$. \square

Remark 3.1. Theorem 1 implies that each second twisted $\frac{U(3)}{U(1) \times U(2)}$ -flow, for a fixed (d_1, d_3) , gives a GMV solution. We will prove that different (d_1, d_3) 's give different second solutions to the same GMV solution (3.31) once $d_3 - d_1$'s are equal (Theorem 6). Thus the GMV equation is only part of the constraints in the second twisted $\frac{U(3)}{U(1) \times U(2)}$ -flows. However, the twisted $\frac{U(3)}{U(1) \times U(2)}$ -hierarchy is still called the (generalized) associated hierarchy for the GMV equation in this paper.

Remark 3.2. The obstruction to constructing the GMV equation parametrized by an arbitrary pair (α, β) by the second twisted $\frac{U(3)}{U(1) \times U(2)}$ -flow is the commutativity condition (2.12).

4 The direct problem

4.1 The spectral problem

Let σ_2, a be defined by (3.6),

$$b = b(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & u & -\bar{v} \\ 0 & v & \bar{u} \end{pmatrix} \in K'_1, \tag{4.1}$$

and $b - 1 \in \mathbb{S}$. Consider the spectral problem

$$\begin{aligned}
\partial_x \Psi &= \lambda b a b^{-1} \Psi + \frac{\epsilon}{\lambda} \sigma_2 (b a b^{-1}) \Psi, \\
\Psi(x, \lambda) e^{-x(\lambda a + \frac{\epsilon}{\lambda} \sigma_2(a))} &\rightarrow 1 \text{ as } x \rightarrow -\infty.
\end{aligned} \tag{4.2}$$

By introducing the normalizations

$$\Psi(x, \lambda) = m(x, \lambda)e^{x(\lambda a + \frac{\epsilon}{\lambda}\sigma_2(a))} \quad (4.3)$$

$$= b(x)m'(x, \lambda)e^{x(\lambda a + \frac{\epsilon}{\lambda}\sigma_2(a))}, \quad (4.4)$$

the partial differential equation in (4.2) turns into

$$\frac{\partial m}{\partial x} = \lambda (bab^{-1}m - ma) + \frac{\epsilon}{\lambda} (\sigma_2(bab^{-1})m - m\sigma_2(a)), \quad (4.5)$$

$$\frac{\partial m'}{\partial x} = [\lambda a + \frac{\epsilon}{\lambda}\sigma_2(a), m'(x, \lambda)] + Q(x, \lambda)m'(x, \lambda), \quad (4.6)$$

with

$$Q(x, \lambda) = \frac{\epsilon}{\lambda} (b^{-1}\sigma_2(bab^{-1})b - \sigma_2(a)) - b^{-1}\frac{\partial b}{\partial x}. \quad (4.7)$$

Definition 2. We define the operator $\mathcal{J}_\lambda = \mathcal{J}_{a, \lambda}$ on $gl(n, \mathbb{C})$ by

$$\mathcal{J}_\lambda f = \left[\lambda a + \frac{\epsilon}{\lambda}\sigma_2(a), f \right],$$

and $\pi_0^\lambda, \pi_\pm^\lambda$ to be the orthogonal projections of $gl(3, \mathbb{C})$ to the $0-, \pm$ -eigenspaces of $Re \mathcal{J}_\lambda = \frac{1}{2}(\mathcal{J}_\lambda + (\mathcal{J}_\lambda)^*)$. Besides, the characteristic curve of (3.6) is defined by

$$\Sigma_a = \{ \lambda \in \mathbb{C} \mid \text{the projections } \pi_0^\lambda, \pi_\pm^\lambda \text{ are not continuous at } \lambda \}. \quad (4.8)$$

A direct computation yields the characteristic curve Σ_a of (3.6) is \mathbb{R} . Therefore we can follow the argument as that in [15] to derive

Theorem 2. Let $b(x) \in K'_1$ and $b - 1 \in \mathbb{S}$. Then there exists a bounded set $Z \subset \mathbb{C}$, such that $Z \cap (\mathbb{C} \setminus \mathbb{R})$ is discrete in $\mathbb{C} \setminus \mathbb{R}$ and for $\forall \lambda \in \mathbb{C} \setminus (\mathbb{R} \cup Z)$, there exists uniquely a solution $m(x, \lambda)$ of (4.5) satisfying

$$m(\cdot, \lambda) \text{ is bounded for each } \lambda \in \mathbb{C} \setminus (\mathbb{R} \cup Z), \quad (4.9)$$

$$m(x, \lambda) \rightarrow 1 \text{ as } x \rightarrow -\infty \text{ for each } \lambda \in \mathbb{C} \setminus (\mathbb{R} \cup Z), \quad (4.10)$$

$$m(x, \cdot) \text{ is meromorphic in } \mathbb{C} \setminus \mathbb{R} \text{ with poles at } \lambda \in Z, \quad (4.11)$$

$$m(x, \lambda) \rightarrow b(x) \text{ uniformly as } |\lambda| \rightarrow \infty. \quad (4.12)$$

Furthermore we have

Theorem 3. For generic $b(x)$ satisfying the assumption of Theorem 2, the set Z is a finite set contained in $\mathbb{C} \setminus \mathbb{R}$, and $m(x, \lambda)$ has a continuous extension, denoted as $m_\pm(x, \lambda)$, to \mathbb{R} from \mathbb{C}^\pm . In addition, there exists $V(\lambda)$, $\lambda \in \mathbb{R} \cup Z$, such that

$$m_+(x, \lambda) = m_-(x, \lambda)e^{x(\lambda a + \frac{\epsilon}{\lambda}\sigma_2(a))}V(\lambda)e^{-x(\lambda a + \frac{\epsilon}{\lambda}\sigma_2(a))}, \quad \lambda \in \mathbb{R}, \quad (4.13)$$

$$m(x, \lambda) \left(1 - \frac{e^{x(\lambda a + \frac{\epsilon}{\lambda}\sigma_2(a))}V(\lambda_0)e^{-x(\lambda a + \frac{\epsilon}{\lambda}\sigma_2(a))}}{\lambda - \lambda_0} \right) \text{ is regular at } \lambda_0 \in Z, \quad (4.14)$$

and for $\lambda \in \mathbb{R}$

$$\partial_\lambda^\alpha (V - 1) \text{ is } \mathcal{O}(\lambda^N) \text{ as } \lambda \rightarrow 0 \text{ and } \mathcal{O}(\lambda^{-N}) \text{ as } |\lambda| \rightarrow \infty \quad (4.15)$$

for all positive integer N and nonnegative integer α ,

$$\det V \equiv 1, \quad (4.16)$$

$$V(\bar{\lambda})^* V(\lambda)^{-1} = 1, \quad (4.17)$$

$$\sigma_1(V(-\lambda))V(\lambda) = 1, \quad (4.18)$$

$$\sigma_2(V(\epsilon/\lambda))V(\lambda) = 1, \quad (4.19)$$

and for $\lambda \in Z$,

$$V(\lambda)^2 = 0, \quad (4.20)$$

$$V(\lambda) = -V(\bar{\lambda})^*, \quad (4.21)$$

$$V(\lambda) = -\sigma_1(V(-\lambda)), \quad (4.22)$$

$$V(\lambda) = -\frac{\lambda^2}{\epsilon} \sigma_2(V(\frac{\epsilon}{\lambda})). \quad (4.23)$$

Proof. The generic property for simple pole with residue satisfying (4.20) has been shown in [3]. The reality conditions (4.17)-(4.19), (4.21)-(4.23) can be proved by showing

$$m(x, \bar{\lambda})^* = m(x, \lambda)^{-1}, \quad (4.24)$$

$$\sigma_1(m(x, -\lambda)) = m(x, \lambda), \quad (4.25)$$

$$\sigma_2(m(x, \epsilon/\lambda)) = m(x, \lambda), \quad (4.26)$$

and using the properties (4.13), (4.14), and (4.20). Finally, by the same argument as that in the proof of Proposition 2.1 in [3], one can prove

$$\det \Psi = \det m \equiv 1. \quad (4.27)$$

The statement (4.16) follows from (4.27) and (4.13). \square

Definition 3. *The associated scattering data of the generic potential $b(x)$ is defined by the matrix function $V(\lambda)$, $\lambda \in \mathbb{R} \cup Z$, provided b satisfies the assumption of Theorem 3. Moreover, $V(\lambda)$ is called a scattering data, if $V(\lambda)$, $\lambda \in \mathbb{R} \cup Z$, satisfies (4.15)-(4.23).*

4.2 An extended direct problem

We need to consider an extended spectral problem of (4.2) for solving the inverse problem. The first criteria for an extended spectral problem is preserving the reality conditions with respect to involutions σ_1 , σ_2 and the self-adjointness. So there could

be multiple choices for extended spectral problems. We choose a (splitting type) twisted $\frac{U(4)}{U(2) \times U(2)}$ spectral problem to be our extended system. Since the inverse scattering problem of twisted $\frac{U(4)}{U(2) \times U(2)}$ -flows is almost the same as that of the twisted $\frac{O(n,n)}{O(n) \times O(n)}$ which has been tackled in [1], [15].

Let $\tilde{\sigma}_i$, $i = 1, 2$, be involutions on $U(4)$ defined by

$$\begin{aligned} \tilde{\sigma}_i(x) &= \tilde{J}_i x \tilde{J}_i^{-1}, \quad x \in U(4), \\ \tilde{J}_1 &= \text{diag}(1, 1, -1, -1), \quad \tilde{J}_2 = \text{diag}(1, 1, -1, 1) \end{aligned} \quad (4.28)$$

and $u(4) = \tilde{K}_i \oplus \tilde{\mathcal{P}}_i$, $i = 1, 2$, the Cartan decompositions for $\tilde{\sigma}_i$. Let \tilde{K}_i be the Lie algebras of \tilde{K}_i , i.e.,

$$\begin{aligned} \tilde{K}_1 &= \left\{ \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{pmatrix} \in U(4) : \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \begin{pmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{pmatrix} \in U(2) \right\}, \\ \tilde{K}_2 &= \left\{ \begin{pmatrix} a_{11} & a_{12} & 0 & a_{14} \\ a_{21} & a_{22} & 0 & a_{24} \\ 0 & 0 & a_{33} & 0 \\ a_{41} & a_{42} & 0 & a_{44} \end{pmatrix} \in U(4) : |a_{33}| = 1, \begin{pmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{41} & a_{42} & a_{44} \end{pmatrix} \in U(3) \right\}, \\ \tilde{\mathcal{P}}_1 &= \left\{ i \begin{pmatrix} 0 & 0 & u_1 & v_1 \\ 0 & 0 & u_2 & v_2 \\ u_1^* & u_2^* & 0 & 0 \\ v_1^* & v_2^* & 0 & 0 \end{pmatrix} \in u(4) \right\}, \quad \tilde{\mathcal{P}}_2 = \left\{ i \begin{pmatrix} 0 & 0 & u_1 & 0 \\ 0 & 0 & u_2 & 0 \\ u_1^* & u_2^* & 0 & u_3 \\ 0 & 0 & u_3^* & 0 \end{pmatrix} \in u(4) \right\}. \end{aligned}$$

Let

$$\begin{aligned} \tilde{S} &= \left\{ \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in U(4) : \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in U(2) \right\} \subset \tilde{K}_1 \cap \tilde{K}_2, \\ \tilde{\mathcal{S}} &= \left\{ \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in u(4) : \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in u(2) \right\}, \end{aligned}$$

and

$$\tilde{K}_1 = \tilde{S} \otimes \tilde{K}'_1, \quad \tilde{K}'_1 = 1_{2 \times 2} \otimes U(2), \quad (4.29)$$

$$\tilde{\mathcal{K}}_1 = \tilde{\mathcal{S}} \oplus \tilde{\mathcal{K}}'_1, \quad \tilde{\mathcal{K}}'_1 = 0_{2 \times 2} \oplus u(2). \quad (4.30)$$

The extended spectral problem of (4.2) is

$$\begin{aligned} \partial_x \tilde{\Psi} &= \lambda \tilde{b} \tilde{a}_1 \tilde{b}^{-1} \tilde{\Psi} + \frac{\epsilon}{\lambda} \tilde{\sigma}_2 (\tilde{b} \tilde{a}_1 \tilde{b}^{-1}) \tilde{\Psi}, \\ \tilde{\Psi}(x, \lambda) e^{-x(\lambda \tilde{a}_1 + \frac{\epsilon}{\lambda} \tilde{\sigma}_2(\tilde{a}_1))} &\rightarrow 1 \text{ as } x \rightarrow -\infty, \end{aligned} \quad (4.31)$$

with

$$\tilde{a}_1 = i \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{b}(x, t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & u & -\bar{v} \\ 0 & 0 & v & \bar{u} \end{pmatrix} \in \tilde{K}'_1. \quad (4.32)$$

We note that \tilde{a}_1 is not an oblique direction for solving the twisted $\frac{U(4)}{U(2) \times U(2)}$ -spectral problem (cf [1], [15]). However, for the extended spectral problem (4.31), (4.32), we have

Lemma 4.1. *The spectral equation (4.2) is satisfied by $\Psi(x, \lambda) = (\Psi_{ij})_{1 \leq i, j \leq 3}$ if and only if (4.31) is satisfied by*

$$\tilde{\Psi}(x, \lambda) = \begin{pmatrix} \Psi_{11} & 0 & \Psi_{12} & \Psi_{13} \\ 0 & 1 & 0 & 0 \\ \Psi_{21} & 0 & \Psi_{22} & \Psi_{23} \\ \Psi_{31} & 0 & \Psi_{32} & \Psi_{33} \end{pmatrix}. \quad (4.33)$$

Moreover, let $m = (m_{ij})_{1 \leq i, j \leq 3}$, $m' = (m'_{ij})_{1 \leq i, j \leq 3}$ be the normalized eigenfunctions defined by (4.3) and (4.4) and

$$\tilde{\Psi}(x, \lambda) = \tilde{m}(x, \lambda) e^{x(\lambda \tilde{a}_1 + \frac{\epsilon}{\lambda} \tilde{\sigma}_2(\tilde{a}_1))} \quad (4.34)$$

$$= \tilde{b}(x) \tilde{m}'(x, \lambda) e^{x(\lambda \tilde{a}_1 + \frac{\epsilon}{\lambda} \tilde{\sigma}_2(\tilde{a}_1))}. \quad (4.35)$$

Then

$$\tilde{m}(x, \lambda) = \begin{pmatrix} m_{11} & 0 & m_{12} & m_{13} \\ 0 & 1 & 0 & 0 \\ m_{21} & 0 & m_{22} & m_{23} \\ m_{31} & 0 & m_{32} & m_{33} \end{pmatrix}, \quad \tilde{m}'(x, \lambda) = \begin{pmatrix} m'_{11} & 0 & m'_{12} & m'_{13} \\ 0 & 1 & 0 & 0 \\ m'_{21} & 0 & m'_{22} & m'_{23} \\ m'_{31} & 0 & m'_{32} & m'_{33} \end{pmatrix}. \quad (4.36)$$

Finally, for generic b , there exists a finite set $Z \subset \mathbb{C} \setminus \mathbb{R}$ and

$$\tilde{m}_+(x, \lambda) = \tilde{m}_-(x, \lambda) e^{x(\lambda \tilde{a}_1 + \frac{\epsilon}{\lambda} \tilde{\sigma}_2(\tilde{a}_1))} \tilde{V}(\lambda) e^{-x(\lambda \tilde{a}_1 + \frac{\epsilon}{\lambda} \tilde{\sigma}_2(\tilde{a}_1))}, \quad \lambda \in \mathbb{R}, \quad (4.37)$$

$$\tilde{m}(x, \lambda) \left(1 - \frac{e^{x(\lambda \tilde{a}_1 + \frac{\epsilon}{\lambda} \tilde{\sigma}_2(\tilde{a}_1))} \tilde{V}(\lambda_0) e^{-x(\lambda \tilde{a}_1 + \frac{\epsilon}{\lambda} \tilde{\sigma}_2(\tilde{a}_1))}}{\lambda - \lambda_0} \right) \text{ is regular at } \lambda_0 \in \mathbb{Z}. \quad (4.38)$$

with

$$\tilde{V}(\lambda) = \begin{pmatrix} V_{11} & 0 & V_{12} & V_{13} \\ 0 & 1 & 0 & 0 \\ V_{21} & 0 & V_{22} & V_{23} \\ V_{31} & 0 & V_{32} & V_{33} \end{pmatrix} \quad (4.39)$$

and for $\lambda \in \mathbb{R}$,

$$\partial_\lambda^\alpha (\tilde{V} - 1) \text{ is } \mathcal{O}(\lambda^N) \text{ as } \lambda \rightarrow 0 \text{ and } \mathcal{O}(\lambda^{-N}) \text{ as } |\lambda| \rightarrow \infty \quad (4.40)$$

$$\det \tilde{V} \equiv 1, \quad (4.41)$$

$$\tilde{V}(\bar{\lambda})^* \tilde{V}(\lambda)^{-1} = 1, \quad (4.42)$$

$$\tilde{\sigma}_1(\tilde{V}(-\lambda)) \tilde{V}(\lambda) = 1, \quad (4.43)$$

$$\tilde{\sigma}_2(\tilde{V}(\epsilon/\lambda)) \tilde{V}(\lambda) = 1, \quad (4.44)$$

and for $\lambda \in \mathbb{Z}$,

$$\tilde{V}(\lambda)^2 = 0, \quad (4.45)$$

$$\tilde{V}(\lambda) = -\tilde{V}(\bar{\lambda})^*, \quad (4.46)$$

$$\tilde{V}(\lambda) = -\sigma_1(\tilde{V}(-\lambda)), \quad (4.47)$$

$$\tilde{V}(\lambda) = -\frac{\lambda^2}{\epsilon} \sigma_2(\tilde{V}(\frac{\epsilon}{\lambda})). \quad (4.48)$$

Proof. The statements can be proved by a direct computation. \square

Definition 4. The associated extended scattering data of b is defined by the matrix function $\tilde{V}(\lambda)$, $\lambda \in \mathbb{R} \cup \mathbb{Z}$, provided b satisfies the assumption of Theorem 3. Moreover, $\tilde{V}(\lambda)$ is called an extended scattering data, if $\tilde{V}(\lambda)$, $\lambda \in \mathbb{R} \cup \mathbb{Z}$, satisfies (4.39)-(4.48).

Remark 4.1. If $b(x) \in K'_1$, $b(x) - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \in \mathbb{S}$, then the spectral problem needed to be considered is

$$\begin{aligned} \partial_x \Psi &= \lambda b a b^{-1} \Psi + \frac{\epsilon}{\lambda} \sigma_2(b a b^{-1}) \Psi, \\ \Psi(x, \lambda) e^{-x(\lambda a + \frac{\epsilon}{\lambda} \sigma_2(a))} &\rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \text{ as } x \rightarrow -\infty. \end{aligned} \quad (4.49)$$

It is more convenient to use a change of variables to turn (4.49) into

$$\begin{aligned} \partial_x \Psi &= \lambda b' a' b'^{-1} \Psi + \frac{\epsilon}{\lambda} \sigma_2(b' a' b'^{-1}) \Psi, \\ \Psi(x, \lambda) e^{-x(\lambda a + \frac{\epsilon}{\lambda} \sigma_2(a))} &\rightarrow 1 \text{ as } x \rightarrow -\infty, \end{aligned} \quad (4.50)$$

with

$$a' = i \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \mathcal{P}_1, \quad b'(x, t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \bar{v} & u \\ 0 & -\bar{u} & v \end{pmatrix} \in K'_1. \quad (4.51)$$

By analogy, one can derive the existence theorem of the eigenfunction $\Psi(x, \lambda)$, extract continuous and discrete scattering data, and solve the associated extended direct problem

$$\begin{aligned} \partial_x \tilde{\Psi} &= \lambda \tilde{b}' \tilde{a}_2 \tilde{b}'^{-1} \tilde{\Psi} + \frac{\epsilon}{\lambda} \tilde{\sigma}_2(\tilde{b}' \tilde{a}_2 \tilde{b}'^{-1}) \tilde{\Psi}, \\ \tilde{\Psi}(x, \lambda) e^{-x(\lambda \tilde{a} + \frac{\epsilon}{\lambda} \tilde{\sigma}_2(\tilde{a}))} &\rightarrow 1 \text{ as } x \rightarrow -\infty, \end{aligned} \quad (4.52)$$

with

$$\tilde{a}_2 = i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \tilde{b}'(x, t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \bar{v} & u \\ 0 & 0 & -\bar{u} & v \end{pmatrix} \in \tilde{K}'_1, \quad (4.53)$$

and define the extended scattering data $\tilde{V}(\lambda)$, $\lambda \in \mathbb{R} \cup Z$.

These two different boundary conditions, (4.2) and (4.49), are the only cases which can be tackled by our approach. Since the associated spectral operators are perturbation of diagonalizable operators. That is, a and $\sigma(a)$ (a' and $\sigma(a')$ respectively) can be simultaneously diagonalized.

5 The inverse problem

In this section, the normalized eigenfunction $\tilde{m}'(x, \lambda)$ will be constructed from the scattering data by solving a Riemann-Hilbert problem. To find the gauge $\tilde{b}(x)$ to reconstruct $\tilde{m}(x, \lambda)$, one needs to understand the symmetries between coefficients of $(\partial_x \tilde{m}')(\tilde{m}')^{-1}$ which is equivalent to solving an over-determined differential systems. Inspired by the result of [1], [15], we reconstruct the gauge via solving an exterior differential system derived from one-dimensional systems associated with Cartan subalgebras with higher ranks (cf. Definition 3.1 in [15]). This is the motivation for us to study the extended twisted $\frac{U(4)}{U(2) \times U(2)}$ -spectral problem in § 4.2.

The major differences between loop algebra structures associated with twisted $\frac{O(2,2)}{O(2) \times O(2)}$ - and with twisted $\frac{U(4)}{U(2) \times U(2)}$ -hierarchies are the symmetric and antisymmetric properties of \mathcal{P}_0 and $\tilde{\mathcal{P}}_1$. However, the proof of the inverse problem in Section 6 of [15] mainly involves with the involution properties of σ_i , the commutativity property $[a_i, a_j] = 0$ and the self-adjointness of K_0 , and has nothing to do with the symmetric property of \mathcal{P}_0 . As a result, the inverse scattering problem of twisted $\frac{U(4)}{U(2) \times U(2)}$ -hierarchy can be solved by the same argument. We will state the results, leave analogous details to [1], [15], and only give the proof for projecting the extended inverse results to that of a twisted $\frac{U(3)}{U(1) \times U(2)}$ -spectral problem in this section.

Write $\tilde{a} = \begin{pmatrix} 0 & D \\ -D^* & 0 \end{pmatrix}$, and $D = \text{diag}(w_1, w_2)$. Define

$$\vec{x} = (x_1, x_2) = x(w_1, w_2), \quad (5.1)$$

$$X = x_1 \tilde{a}_1 + x_2 \tilde{a}_2, \quad (5.2)$$

$$\tilde{a}_1 = i \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{a}_2 = i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \quad (5.3)$$

Theorem 4. Let $\tilde{V}(\lambda)$, $\lambda \in \mathbb{R}$ satisfy the analytical constraints (4.40) and the algebraic constraints (4.41)-(4.48). Then there exists uniquely $M(\vec{x}, \lambda)$ such that

$$M_+(\vec{x}, \lambda) = M_-(\vec{x}, \lambda) e^{\lambda X + \frac{\epsilon}{\lambda} \tilde{\sigma}_2(X)} \tilde{V}(\lambda) e^{-\lambda X - \frac{\epsilon}{\lambda} \tilde{\sigma}_2(X)}, \quad \lambda \in \mathbb{R}, \quad (5.4)$$

$$M(\vec{x}, \lambda) \left(1 - \frac{e^{\lambda X + \frac{\epsilon}{\lambda} \tilde{\sigma}_2(X)} \tilde{V}(\lambda_0) e^{-(\lambda X + \frac{\epsilon}{\lambda} \tilde{\sigma}_2(X))}}{\lambda - \lambda_0} \right) \text{ is regular at } \lambda_0 \in Z, \quad (5.5)$$

$$M(\vec{x}, \lambda) \text{ is holomorphic for } \lambda \in \mathbb{C} \setminus (\mathbb{R} \cup Z), \quad M(\vec{x}, \lambda) \rightarrow 1 \text{ as } |\lambda| \rightarrow \infty, \quad (5.6)$$

and $M(\vec{x}, \lambda)$ satisfies the analytical and algebraic conditions

$$\det M \equiv 1, \quad (5.7)$$

$$M(\vec{x}, \bar{\lambda})^* = M(\vec{x}, \lambda)^{-1}, \quad (5.8)$$

$$\tilde{\sigma}_1(M(\vec{x}, -\lambda)) = M(\vec{x}, \lambda), \quad \tilde{\sigma}_2(M^{-1}(\vec{x}, 0) M(\vec{x}, \frac{\epsilon}{\lambda})) = M(\vec{x}, \lambda) \quad (5.9)$$

$$x^k \partial_x^{k'} \lambda^{k'} (M(\vec{x}, \lambda) - 1) \in L^2(\mathbb{R}) \text{ for } \forall k, k' \text{ and tends to 0 uniformly} \quad (5.10)$$

$$\text{as } x \rightarrow -\infty; \exists \delta(\lambda) \text{ diagonal, s.t. } x^k \partial_x^{k'} \lambda^{k'} (M(\vec{x}, \lambda) - \delta(\lambda)) \in L^2(\mathbb{R})$$

$$\text{for } \forall k, k' \text{ and tends to 0 uniformly as } x \rightarrow \infty.$$

Moreover, if $\tilde{V}(\lambda)$ is an extended scattering data, i.e. is of the form (4.39), then

$$M((x, 0), \lambda) = \begin{pmatrix} M_{11} & 0 & M_{12} & M_{13} \\ 0 & 1 & 0 & 0 \\ M_{21} & 0 & M_{22} & M_{23} \\ M_{31} & 0 & M_{32} & M_{33} \end{pmatrix}. \quad (5.11)$$

Proof. The existence of $M(\vec{x}, \lambda)$ satisfying (5.4)-(5.6), and (5.10) can be proved by the same argument as that in the proof of Theorem 5.1 in [15]. Properties (4.41), (4.45) imply $\det M$ is continuous for $\lambda \in \mathbb{C}$. Thus Condition (5.7) is shown by noting $\partial_{\bar{\lambda}} \det M = 0$ for $\lambda \in \mathbb{C}^\pm$ and applying Liouville's theorem. The statements (5.8), (5.9), and (5.11) can be proved by the uniqueness property of $M(\vec{x}, \lambda)$. \square

Defining the asymptotic expansions

$$M(\vec{x}, \lambda) \rightarrow 1 + \sum_{k=1}^{\infty} M_k^\sharp(\vec{x}) \lambda^{-k} \quad \text{as } |\lambda| \rightarrow \infty, \quad (5.12)$$

$$M(\vec{x}, \lambda) \rightarrow \sum_{k=0}^{\infty} M_k^b(\vec{x}) \lambda^k \quad \text{as } |\lambda| \rightarrow 0. \quad (5.13)$$

and applying the same argument as that in the proof of Lemma 6.1 - 6.3, and Theorem 6.1 in [15], one can derive the following four lemmas.

Lemma 5.1. Suppose $M(\vec{x}, \lambda)$ is derived by Theorem 4. Then

$$\frac{\partial M}{\partial x_j} = [\lambda \tilde{a}_j + \frac{\epsilon}{\lambda} \tilde{\sigma}_2(\tilde{a}_j), M] + \frac{\epsilon}{\lambda} (B_j(\vec{x}) - \tilde{\sigma}_2(\tilde{a}_j)) M - C_j(\vec{x}) M, \quad (5.14)$$

with

$$B_j(\vec{x}) \in \tilde{\mathcal{P}}_1 \cap C^\infty, \quad C_j(\vec{x}) \in \tilde{\mathcal{K}}_1 \cap C^\infty. \quad (5.15)$$

Lemma 5.2. *The compatibility conditions of (5.14) are*

$$\partial_{x_j} C_i - \partial_{x_i} C_j - [C_i, C_j] = \epsilon [\tilde{a}_i, B_j] - \epsilon [\tilde{a}_j, B_i]. \quad (5.16)$$

Lemma 5.3. *For any constant $\mu \in \mathbb{R}$, there exists uniquely*

$$\tilde{b}(\vec{x}) \in \tilde{K}'_1 \cap C^\infty, \quad (5.17)$$

such that

$$\tilde{b}(x(w_1, w_2)) \rightarrow \begin{pmatrix} 1_{2 \times 2} & 0 \\ 0 & e^{-i\mu/2} 1_{2 \times 2} \end{pmatrix} \in \tilde{K}'_1 \quad \text{as } x \rightarrow -\infty, \quad (5.18)$$

$$-\tilde{b} C_j \tilde{b}^{-1} + (\partial_j \tilde{b}) \tilde{b}^{-1} \in \tilde{\mathcal{S}} \quad \text{for } \forall j. \quad (5.19)$$

Proof. We remark the boundary condition (5.18) can be chosen for arbitrary element in \tilde{K}'_1 . \square

Lemma 5.4. *Suppose the assumption of Theorem 4 holds. For any constant $\mu \in \mathbb{R}$, let*

$$\tilde{\Psi}(x, \lambda) = \tilde{b}(\vec{x}) M(\vec{x}, \lambda) e^{\lambda X + \frac{\epsilon}{\lambda} \sigma_2(X)} \quad (5.20)$$

Here x, \vec{x}, X, M satisfy (5.1)-(5.3). Then

$$\frac{\partial \tilde{\Psi}}{\partial x} = \lambda \tilde{b} \tilde{a} \tilde{b}^{-1} \tilde{\Psi} + \frac{\epsilon}{\lambda} \tilde{\sigma}_2(\tilde{b} \tilde{a} \tilde{b}^{-1}) \tilde{\Psi} + \tilde{v} \tilde{\Psi}, \quad (5.21)$$

with

$$\tilde{v}(\vec{x}) = \sum_{j=1}^2 w_j (-\tilde{b} C_j \tilde{b}^{-1} + (\partial_{x_j} \tilde{b}) \tilde{b}^{-1}) \in \tilde{\mathcal{S}} \cap \mathbb{S}, \quad (5.22)$$

where $C_j, \tilde{b}(\vec{x})$ are defined by Lemma 5.1, 5.3, respectively. Moreover,

$$\tilde{b}(\vec{x}) - \begin{pmatrix} 1_{2 \times 2} & 0 \\ 0 & e^{-i\mu/2} 1_{2 \times 2} \end{pmatrix} \in \tilde{K}'_1 \cap \mathbb{S}, \quad (5.23)$$

$$\tilde{v} \text{ is independent of } \mu \text{ defined by (5.18)}. \quad (5.24)$$

Proof. Once the boundary condition (5.18) is a diagonal element in \tilde{K}'_1 , the argument in proving Theorem 6.1 in [15] works well in proving all statements in Lemma 5.4 except (5.24). The property (5.24) follows from the fact that changing μ could only alter the \tilde{K}'_1 part of the right hand side of (5.22) and $\tilde{v} \in \tilde{\mathcal{S}}$.

Note (5.21) and (5.20) imply

$$\frac{\partial M}{\partial x} = [\lambda \tilde{a} + \frac{\epsilon}{\lambda} \tilde{\sigma}_2(\tilde{a}), M(x, \lambda)] + Q(x, \lambda) M(x, \lambda) \quad (5.25)$$

$$Q(x, \lambda) = \frac{\epsilon}{\lambda} \left(\tilde{b}^{-1} \tilde{\sigma}_2(\tilde{b} \tilde{a} \tilde{b}^{-1}) \tilde{b} - \tilde{\sigma}_2(\tilde{a}) \right) - \tilde{b}^{-1} \frac{\partial \tilde{b}}{\partial x} + \tilde{b}^{-1} \tilde{v} \tilde{b}. \quad (5.26)$$

Thus the Schwartz properties (5.22) and (5.23) follow from (5.10), (5.25), and (5.26). \square

Lemma 5.4 solves the inverse problem of a general twisted $\frac{U(4)}{U(2) \times U(2)}$ -spectral problem for scattering data $\tilde{V}(\lambda)$ satisfying (4.40)-(4.48). The following theorem says that when $\tilde{V}(\lambda)$ is an extended scattering data, the above result can be projected to be a solvability of the inverse problem for a twisted $\frac{U(3)}{U(1) \times U(2)}$ -spectral problem.

Theorem 5. *Suppose the assumption of Theorem 4 holds for either*

$$\tilde{V}(\lambda) = \begin{pmatrix} V_{11} & 0 & V_{12} & V_{13} \\ 0 & 1 & 0 & 0 \\ V_{21} & 0 & V_{22} & V_{23} \\ V_{31} & 0 & V_{32} & V_{33} \end{pmatrix}, \quad (5.27)$$

or

$$\tilde{V}(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & V_{11} & V_{12} & V_{13} \\ 0 & V_{21} & V_{22} & V_{23} \\ 0 & V_{31} & V_{32} & V_{33} \end{pmatrix}. \quad (5.28)$$

Then there exist a unique $\Psi(x, \lambda) \in L_+^\epsilon$ and a unique $b(x) \in K_1'$ satisfying $b(x) - 1 \in \mathbb{S}$ (or $b(x) - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \in \mathbb{S}$) such that

$$\frac{\partial \Psi}{\partial x} = \lambda b a b^{-1} \Psi + \frac{\epsilon}{\lambda} \sigma_2 (b a b^{-1}) \Psi \quad (5.29)$$

with a defined by (3.6), and the associated extended scattering data of $b(x)$ is $\tilde{V}(\lambda)$.

Proof. We only prove the case (5.27). Case (5.28) can be argued by analogy. Let $M(\vec{x}, \lambda)$, $\tilde{b}(\vec{x})$, $\tilde{\Psi}(x, \lambda)$ be derived from Theorem 4, Lemma 5.3 and 5.4 by specially choosing $(w_1, w_2) = (1, 0)$ in (5.1), i.e. $\vec{x} = (x, 0)$, $X = x\tilde{a}_1$, $\tilde{a} = \tilde{a}_1 \in \tilde{\mathcal{A}}$. Define

$$M(\vec{x}, \lambda) = \tilde{m}'(x, \lambda) \quad (5.30)$$

Applying Theorem 4 and (5.27), we have

$$\tilde{m}'(x, \lambda) = \begin{pmatrix} m'_{11} & 0 & m'_{12} & m'_{13} \\ 0 & 1 & 0 & 0 \\ m'_{21} & 0 & m'_{22} & m'_{23} \\ m'_{31} & 0 & m'_{32} & m'_{33} \end{pmatrix}.$$

Together with $\tilde{b} \in \tilde{K}_1'$, we find \tilde{v} in (5.21) is of the form $\text{diag}(i\nu, 0, 0, 0)$. Hence (5.21) can be gauged to

$$\frac{\partial \tilde{\Psi}_1}{\partial x} = \lambda \tilde{b}_1 \tilde{a} \tilde{b}_1^{-1} \tilde{\Psi}_1 + \frac{\epsilon}{\lambda} \tilde{\sigma}_2 (\tilde{b}_1 \tilde{a} \tilde{b}_1^{-1}) \tilde{\Psi}_1, \quad (5.31)$$

$$\tilde{b}_1 = \tilde{b} \cdot \text{diag}(1, 1, e^{i\nu}, 1) \in \tilde{K}_1' = 1 \otimes U(2), \quad (5.32)$$

$$\tilde{\psi}_1 = \text{diag}(e^{i\nu}, 1, 1, 1) \tilde{\Psi}. \quad (5.33)$$

Applying the same argument as that in Proposition 2.1 in [15] to (5.31), we obtain that $\det(\tilde{\Psi}_1)$ is constant. Together with (5.7), (5.20), and (5.33), we conclude $e^{i\nu} \det \tilde{b}$ is constant which equals to $e^{i(\nu(x=-\infty)-\mu)}$ by (5.18) and (5.33). Equation (5.32) then yields

$$\det \tilde{b}_1 = e^{i\nu} \det \tilde{b} \equiv e^{i(\nu(x=-\infty)-\mu)}.$$

Besides, noting $\nu(x = -\infty)$ exists (by (5.22)) and is determined by \tilde{v} (independent of μ). Consequently

$$\tilde{b}_1 = \begin{pmatrix} 1_{2 \times 2} & 0 \\ 0 & \omega \end{pmatrix}, \quad \omega \in SU(2), \quad \tilde{b}_1 - 1 \in \mathbb{S} \quad (5.34)$$

by choosing $\mu = \nu(x = -\infty)$ and using (5.24), (5.32).

However, by solving the direct problem of (5.31) with \tilde{b}_1 satisfying (5.34) and applying (4.12), (5.6), (5.32), and (5.33), as matter of fact, $\nu = 0$.

Therefore the theorem is proved by defining

$$\Psi(x, \lambda) = b(x)m'(x, \lambda)e^{x(\lambda a + \frac{\epsilon}{\lambda}\sigma_2(a))} \quad (5.35)$$

with a defined by (3.6), $m'(x, \lambda) = (m'_{ij})$, and $b = \begin{pmatrix} 1_{1 \times 1} & 0 \\ 0 & \omega \end{pmatrix} \in K'_1$.

□

6 The Cauchy problem

We first apply the inverse scattering theory established in Section 4 and 5 to solve the initial value problem of the k -th twisted $\frac{U(3)}{U(1) \times U(2)}$ -flow.

Theorem 6. *Given $d_1, d_3 \in \mathbb{R}$, and $b_0(x) \in K'_1$ such that either $b_0 - 1 \in \mathbb{S}$ or $b_0 - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \in \mathbb{S}$. If the scattering data for $b_0(x)$ is generic, then the initial value problem of the k -th twisted $\frac{U(3)}{U(1) \times U(2)}$ -flow admits a unique solution $b(x, t) \in \mathfrak{P}_1$ or \mathfrak{P}_2 . More precisely, there uniquely exist $m(x, t, \lambda) \in L_-^\epsilon$ and $\Psi(x, t, \lambda) = m(x, t, \lambda)e^{x\hat{J}_{1,0} + t\hat{J}_k} \in L_+^\epsilon$ such that*

$$[\mathbf{L}, \mathbf{M}] = 0 \quad (6.1)$$

with

$$\begin{aligned} \mathbf{L} &= \partial_x - \frac{\partial \Psi}{\partial x} \Psi^{-1} = \partial_x - (\lambda bab^{-1} + \frac{\epsilon}{\lambda} \sigma_2(bab^{-1})), \\ \mathbf{M} &= \partial_t - \frac{\partial \Psi}{\partial t} \Psi^{-1}, \\ b(x, 0) &= b_0(x), \quad b(x, t) = m(x, t, \infty) \in \mathfrak{P}_1 \text{ (or } b(x, t) \in \mathfrak{P}_2 \text{)}. \end{aligned} \quad (6.2)$$

Proof. We first apply Theorem 2 to solve the eigenfunction of (4.2) for $a, b(x, 0) = b_0(x)$ defined by (3.6) and (4.1). Applying Definition 3 and Theorem 3, we obtain the scattering data $V(\lambda, 0), \lambda \in \mathbb{R} \cup Z$ for the potential $b(x, 0)$. Define

$$V(\lambda, t) = e^{t\hat{J}_k} V(\lambda, 0) e^{-t\hat{J}_k}, \text{ for } \lambda \in \mathbb{R} \cup Z \quad (6.3)$$

So $V(\lambda, t)$ satisfies the assumption of Theorem 5 and there exist uniquely smooth $m'(x, t, \lambda) \in L_-^\epsilon, b(x, t) \in \mathfrak{P}_1$ (or $b(x, t) \in \mathfrak{P}_2$) satisfying

$$m(x, t, \lambda) = b(x, t) m'(x, t, \lambda), \quad (6.4)$$

$$\Psi(x, t, \lambda) = b(x, t) m'(x, t, \lambda) e^{x\hat{J}_{1,0} + t\hat{J}_k}, \quad (6.5)$$

and

$$\frac{\partial \Psi}{\partial x}(x, t, \lambda) = \lambda b a b^{-1} \Psi(x, t, \lambda) + \frac{\epsilon}{\lambda} \sigma_2 (b a b^{-1}) \Psi(x, t, \lambda). \quad (6.6)$$

So $\mathbf{L} = \frac{\partial \Psi}{\partial x} \Psi^{-1} \in \mathcal{L}_+^\epsilon$. Therefore, $\mathbf{M} = \frac{\partial \Psi}{\partial t} \Psi^{-1} \in \mathcal{L}_+^\epsilon$ and $[\mathbf{L}, \mathbf{M}] = 0$. \square

Consequently, we can solve the initial value problem for the GMV equation for $(\alpha, \beta) = (\alpha, -4\epsilon)$ or $(\alpha, \beta) = (4\epsilon, \beta)$.

Corollary 6.1. *Given $\epsilon > 0, \beta \in \mathbb{R}$, and a (generic) function $\vec{u}_0(x) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{S}$, the initial value problem of the GMV equation*

$$\begin{aligned} i\vec{u}_t &= (\vec{u}_x - \vec{u}(\vec{u}^* \cdot \vec{u}_x))_x + 4\epsilon \vec{u}(\vec{u}^* \cdot J\vec{u}) + \mathbf{A}\vec{u}, \\ \vec{u}^* \vec{u} &= 1, \quad \vec{u} \in \mathbb{C}^2, \quad \vec{u}(x, 0) = \vec{u}_0(x), \\ J &= \text{diag}(-1, 1), \quad \mathbf{A} = \text{diag}(4\epsilon, \beta), \end{aligned} \quad (6.7)$$

admits one family of global solutions.

Proof. The solvability follows from setting \hat{J}_k to be

$$\hat{J}_2 = i(a^2 \lambda^2 - \begin{pmatrix} 2\epsilon + \alpha_1 & 0 & 0 \\ 0 & 2\epsilon + \alpha_1 & 0 \\ 0 & 0 & \beta + \alpha_1 \end{pmatrix} + \sigma_2(a^2)(\frac{\epsilon}{\lambda})^2), \quad \alpha_1 \in \mathbb{R},$$

and applying Theorem 1 and 6. Different α_1 's correspond to different $b(x, t)$'s since the scattering data differ when $t > 0$ by (6.3). \square

Corollary 6.2. *Given $\epsilon > 0, \alpha \in \mathbb{R}$, and a (generic) function $\vec{u}_0(x) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{S}$, the initial value problem of the GMV equation*

$$\begin{aligned} i\vec{u}_t &= (\vec{u}_x - \vec{u}(\vec{u}^* \cdot \vec{u}_x))_x + 4\epsilon \vec{u}(\vec{u}^* \cdot J\vec{u}) + \mathbf{A}\vec{u}, \\ \vec{u}^* \vec{u} &= 1, \quad \vec{u} \in \mathbb{C}^2, \quad \vec{u}(x, 0) = \vec{u}_0(x), \\ J &= \text{diag}(-1, 1), \quad \mathbf{A} = \text{diag}(\alpha, -4\epsilon), \end{aligned} \quad (6.8)$$

admits one family of global solutions.

Proof. Set \hat{J}_k to be

$$\hat{J}_2 = i(a^2\lambda^2 - \begin{pmatrix} -2\epsilon + \alpha_1 & 0 & 0 \\ 0 & -2\epsilon + \alpha_1 & 0 \\ 0 & 0 & \alpha + \alpha_1 \end{pmatrix} + \sigma_2(a^2)(\frac{\epsilon}{\lambda})^2), \quad \alpha_1 \in \mathbb{R},$$

and apply Theorem 1 and 6. Different α_1 's correspond to different $b(x, t)$'s since the scattering data differ when $t > 0$ by (6.3). \square

References

- [1] M. Ablowitz, R. Beals, K. Tenenblat: On the solution of the generalized wave and generalized sine Gordon equations. *Stud. Appl. Math.* 74 (1986), no. 3, 177–203.
- [2] M. Adler: On a trace functional for formal pseudo differential operators and the symplectic structure of the Korteweg-de Vries type equations, *Invent. Math.*, 50 (1978/79), no. 3, 219–248.
- [3] R. Beals and R. R. Coifman: Scattering and inverse scattering for first order systems, *Comm. Pure Appl. Math.* **37** (1984), no. 1, 39–90.
- [4] R. Beals, K. Tenenblat: Inverse scattering and the Bäcklund transformation for the generalized wave and generalized sine-Gordon equations. *Stud. Appl. Math.* 78 (1988), no. 3, 227–256.
- [5] R. K. Dodd: Classification of integrable equations. Integrable and superintegrable systems, *World Sci. Publ., Teaneck, NJ*, (1990), 102–133.
- [6] V. G. Drinfel'd, V. V. Sokolov: Lie algebras and equations of Korteweg-de Vries type. (Russian) Current problems in mathematics, Vol. 24, 81–180, *Itogi Nauki i Tekhniki*, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1984.
- [7] V. S. Gerdjikov, G. G. Grahovski, R. I. Ivanov, N. A. Kostov: N -wave interactions related to simple Lie algebras. Z_2 -reductions and soliton solutions. *Inverse Problems* **17**, 4, (2001), 999–1015.
- [8] V. S. Gerdjikov, G. G. Grahovski, N. A. Kostov: Reductions of N -waves interactions related to low rank simple Lie algebras. I. Z_2 -reductions. *J. Phys. A: Math. Gen.* **34** (2001), 9425–9461.
- [9] V. S. Gerdjikov, G. G. Grahovski, A. V. Mikhailov, T. I. Valchev: On Soliton Interactions for a Hierarchy of Generalized Heisenberg Ferromagnetic Models on $SU(3)/S(U(1) \times U(2))$ Symmetric Space. arXiv:1201.0534

- [10] V. S. Gerdjikov, A. V. Mikhailov, T. I. Valchev: Reductions of integrable equations on **A.III**-type symmetric spaces. *J. Phys. A: Math. Gen.* **43** (2010), no. 43, 434015, 13 pp.
- [11] V. S. Gerdjikov, A. V. Mikhailov, T. I. Valchev: Recursion operators and reductions of integrable equations on symmetric spaces. *J. Geom. Symmetry Phys.* **20** (2010), 1–34.
- [12] S. Lombardo, A. V. Mikhailov: Reductions of integrable equations: Dihedral group. *J. Phys. A: Math. Gen.* **37** (2004), 7727–7742.
- [13] S. Lombardo, A. V. Mikhailov: Reductions groups and automorphic Lie algebras *Commun. Math. Phys.* **258** (2005), 179–202.
- [14] S. Lombardo, Sara; J. A. Sanders: On the classification of automorphic Lie algebras. *Comm. Math. Phys.* **299** (2010), no. 3, 793–824.
- [15] H. Ma, D. Wu: Twisted hierarchies associated with the generalized sine-Gordon equation. *Journal of Mathematical Physics* **52** (2011), 093704, 33 pp.
- [16] A. V. Mikhailov: On the integrability of two-dimensional generalization of the Toda Lattice. *Lett. in Jour. of Experimental and Theoretical Physics* **30** (1979), 443–448.
- [17] A. V. Mikhailov: The reduction problem and the inverse scattering method. in Soliton Theory, proceedings of the Soviet-American symposium on Soliton Theory, Kiev, USSR, *Physica 3D*, 1&2 (1981), 73–117.
- [18] A. V. Mikhailov: The Landau-Lifschitz equation and the Riemann boundary problem on a torus. *Physics Letters A* **92**, (1982), 51–55.
- [19] A. V. Mikhailov, A. B. Shabat, V. V. Sokolov: The symmetry approach to classification of integrable equations. What is integrability? *Springer Ser. Nonlinear Dynam.*, Springer, Berlin, (1991), 115–184.
- [20] A. V. Mikhailov, A. B. Shabat, R. I. Yamilov: Extension of the module of invertible transformations. Classification of integrable systems. *Comm. Math. Phys.* **115** (1988), no. 1, 1–19.
- [21] C. L. Terng: Geometries and symmetries of soliton equations and integrable elliptic equations. Surveys on geometry and integrable systems, *Adv. Stud. Pure Math.*, **51**, Math. Soc. Japan, Tokyo, (2008), 401–488.
- [22] C. L. Terng: Soliton hierarchies constructed from involutions. *Fourth International Congress of Chinese Mathematicians*, 367–381, AMS/IP Stud. Adv. Math., **48**, Amer. Math. Soc., Providence, RI, 2010.

- [23] C. L. Terng, K. Uhlenbeck: Bäcklund transformations and loop group actions, *Comm. Pure Appl. Math.* 53 (2000), no. 1, 1–75.
- [24] G. Wilson: The τ -functions of the $gAKNS$ equations. Integrable systems (Luminy, 1991), 131–145, *Progr. Math.*, 115, Birkhäuser Boston, Boston, MA, 1993.